Test 3 Numerical Mathematics 2
November, 2020

Duration: 5 quarters of an hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. Consider the class of integrals: $\int_0^{\pi/2} \sin(x) f(x) dx$.
   (a) [2.5] Show that the first Gauss rule for an integral of this type is $f(1)$.
   (b) [0.5] Determine the exact and the approximate integral of the polynomials $1, x$, and x^2. Deduce from that the degree of exactness of the rule.

2. Consider the function
$$f(x) = \begin{cases} 2x & \text{for } x = [0, 1/2], \\ 1 & \text{for } x = [1/2, 1]. \end{cases}$$
   (a) [1] Make a sketch of $f(x)$ and show that the best linear approximation on [0,1] is given by $x + 1/4$. Make also also a sketch of the error. Which property is relevant for the proof?
   (b) [2] Give the linear approximation resulting from an interpolation on zeros of an appropriate Chebyshev polynomial. Explain why taking the zeros of a Chebyshev polynomial as interpolation points is a reasonable choice.

3. On [-1,1], consider the function
$$f(x) = \begin{cases} 1 + x & \text{for } x > 0, \\ -1 + x & \text{for } x \leq 0. \end{cases}$$
   Suppose the least-squares approximation of the function $f(x)$ is given by $\sum_{i=0}^{\infty} \alpha_i \phi_i(x)$ where $\phi_i(x)$ are orthogonal polynomials with respect to the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)dx$.
   (a) [1.5] Show that $\alpha_i = 0$ for i is even.
   (b) [1.5] Compute α_1.

Consider the class of integrals $\int_0^{\frac{1}{2}\pi} \sin(x) f(x)\, dx$

Derive the first Gauss rule for this type of integrals

Consider $(f, g) \equiv \int_0^{\frac{1}{2}\pi} \sin(x)\, f\, g\, dx$

we need the first order orth. polynomial to give us the interpolation point

$\varphi_0 = 1, \quad \varphi_1 = x - \alpha$

$(\varphi_1, \varphi_0) = 0 \Rightarrow \int_0^{\frac{1}{2}\pi} \sin(x)(x - \alpha)\, dx = 0$

$$\alpha = \frac{\int_0^{\frac{1}{2}\pi} x \sin(x)\, dx}{\int_0^{\frac{1}{2}\pi} \sin x\, dx}$$

$\int_0^{\frac{1}{2}\pi} \sin x\, dx = -\cos x \Big|_0^{\frac{1}{2}\pi} = 1$

$\int_0^{\frac{1}{2}\pi} x \sin x\, dx = -x \cos x \Big|_0^{\frac{1}{2}\pi} + \int_0^{\frac{1}{2}\pi} \cos x\, dx =$

$= \sin x \Big|_0^{\frac{1}{2}\pi} = 1$

$\therefore \alpha = 1$ it is also the sought zero.

Rule $\int_0^{\frac{1}{2}\pi} \sin x\, dx = f(1)$

b Integrate $1, x, x^2$

	Exact		Rule
1:	$\int_0^{\frac{1}{2}\pi} \sin x\, dx = 1$	$\xleftrightarrow{\text{equal}}$	1
x:	$\int_0^{\frac{1}{2}\pi} x \sin x\, dx = 1$	$\xleftrightarrow{\text{equal}}$	1
x^2:	$\int_0^{\frac{1}{2}\pi} x^2 \sin x\, dx = -x^2\cos x \Big|_0^{\frac{1}{2}\pi} - 2\int_0^{\frac{1}{2}\pi} x \cos x\, dx = -2\, x \sin x \Big|_0^{\frac{1}{2}\pi} + 2\int_0^{\frac{1}{2}\pi} \cos x = 2$		1

degree 1

not equal

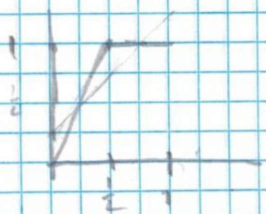
Consider the function

$$f(x) = \begin{cases} 2x & x = 0, \frac{1}{2} \\ 1 & x = \frac{1}{2}, 1 \end{cases}$$

a Make a sketch of $f(x)$ and show that best linear approximation is given by $\frac{1}{4} + x$

which property do you use to determine that?

<u>Answer</u>



$l(x) = a\,x + b$

Equiripple property : extreme error equal and alternating in sign.

$l(0) - f(0) = b - 0 = b$

$l(\frac{1}{2}) - f(\frac{1}{2}) = \frac{1}{2}a + b - 1$

$l(1) - f(1) = a + b - 1$

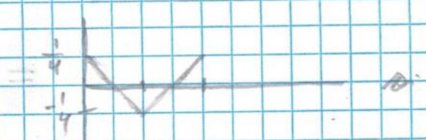
$l(0) - f(0) = -\left(l(\frac{1}{2}) - f(\frac{1}{2})\right) \Rightarrow \frac{1}{2}a + 2b = 1$

$l(1) - f(1) = l(0) - f(0) \Rightarrow a = 1$

$2b = 1 - \frac{1}{2} = \frac{1}{2}$

$b = \frac{1}{4}$

Error plot =



Error satisfies equiripple property so $\frac{1}{4} + x$ is the best lin approximation.

b Give the linear approximation based on zero of a Chebyshev polynomial.

(You may refrain from simplifying the resulting polynomial)

Answer: We need 2 zeroes of a Chebyshev polynomial. So of $T_2(x)$

They follow from the property

$$T_2(\cos\theta) = \cos 2\theta \Rightarrow \cos 2\theta = 0$$

$$\Rightarrow \theta = \tfrac{1}{4}\pi, \tfrac{3}{4}\pi$$

$$x_0 = \cos\tfrac{3}{4}\pi = -\tfrac{1}{2}\sqrt{2}$$

$$x_1 = \qquad = +\tfrac{1}{2}\sqrt{2}$$

Next we need to transform this to the interval [0,1]

$\tilde{x} = \frac{1}{2} + x/2$ should do the trick

$$\tilde{x}_0 = \tfrac{1}{2} - \tfrac{1}{4}\sqrt{2}$$

$$\tilde{x}_1 = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2}$$

The linear interpolation is now

$$l_c(x) = f(\tilde{x}_0)\frac{x-\tilde{x}_1}{\tilde{x}_0-\tilde{x}_1} + f(\tilde{x}_1)\frac{x-\tilde{x}_0}{\tilde{x}_1-\tilde{x}_0}$$

$$= 2\left(\tfrac{1}{2}-\tfrac{1}{4}\sqrt{2}\right)\frac{x-\left(\frac{1}{2}+\frac{1}{4}\sqrt{2}\right)}{-\frac{1}{2}\sqrt{2}} + 1\,\frac{x-\left(\frac{1}{2}-\frac{1}{4}\sqrt{2}\right)}{\frac{1}{2}\sqrt{2}}$$

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On $-1, 1$ we consider

$$f(x) = \begin{cases} 1 + x & x > 0 \\ -1 + x & x \leq 0 \end{cases}$$

Sketch $f(x)$ and

a Show that $a_k = 0$ for k even

in a Legendre expansion.

<u>answer</u>

So $f(x)$ is odd: $f(x) = -f(-x)$

The Legendre polynomials L_k are even for k is even and odd for k is odd so

$$a_k = \frac{(L_k, f)}{(L_k, L_k)} \qquad \text{where } (f,g) \equiv \int_{-1}^{1} f g \, dx$$

now $(L_k, f) = \int_{-1}^{1} L_k \underset{\text{odd}}{f} \, dx = 0 \;\; \forall \; L_k$ even so k even.

$\rightarrow a_k = 0$ for k even.

Compute a_1

$$a_1 = \frac{(L_1, f)}{(L_1, L_1)} \overset{L_1 = x}{=} \frac{(x, f)}{(x, x)} = \frac{2\int_0^1 x(1+x)\,dx}{\int_{-1}^1 x^2\,dx \quad \text{or } 2\int_0^1 x^2\,dx} =$$

$$= \frac{\cancel{2}\left(\frac{1}{2}x^2 + \frac{1}{3}x^3\right)\Big|_0^1}{\cancel{2}\,\frac{1}{3}x^3\Big|_0^1} = \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{3}} = \frac{3}{2} + 1 = 2\frac{1}{2}$$